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# OBSERVATIONS ON THE THEORETICAL AND EXPERIMENTAL FOUNDATIONS OF THERMOPLASTICITY†

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**Abstract**—The thermodynamics of materials with internal state variables has been employed to study the properties of a class of thermoplastic materials in which the evolution equation for the internal variables is given by equation

$$\dot{k}^{(i)} = g^{(i)}(\sigma_{kl}, \epsilon''_{kl}, k^{(i)}, \theta, \dot{\epsilon}''_{kl})$$

where $g^{(i)}$ is homogeneous of order one in $\dot{\epsilon}''_{kl}$. The most general form of the Helmholtz potential consistent with the assumption of insensitivity of the elastic relations to inelastic deformation has been derived and a geometric interpretation of the Clausius-Duhem restriction has been made employing the concept of a thermodynamic reference stress. Experimental results of one of the authors have been correlated with the theory.

## 1. INTRODUCTION

In this paper, some aspects of the theory of thermoplasticity discussed earlier by Phillips and Eisenberg[1] and Phillips[2] are developed further and interpreted in the light of recent experimental work reported by Phillips and Tang[3].

In Section 2 we present a thermodynamic analysis of a plasticity theory which postulates the existence of a finite number of measures of the effects of prior deformation history upon the material response. The analysis parallels the Coleman–Gurtin approach[4] to the thermodynamics of materials with internal state variables.‖ The form of the dependence of the specific Helmholtz free energy $\psi$ on the deformation and thermal history of the material is explored in detail. In particular, it is demonstrated that the Clausius–Duhem inequality and other rather general phenomenological assumptions restrict the form of this dependence, and that there exists a thermodynamic reference stress $\sigma^0_{kl}$, related to $\psi$ which plays a central role in the description of these restrictions. In this section it is shown that $\sigma^0_{kl}$ may depend in a complex way upon the mechanical state of the material, that the stress state will not in general be limited to the interior or boundary of the elastic domain, and that the yield surface in stress space will, under appropriate circumstances, approach a steady state yield surface.

In Section 3 a simplified theory is presented and a method of experimental evaluation of $\sigma^0_{kl}$ is discussed. We restrict our considerations to the analysis of monotonically increasing radial loadings. After introducing the concepts of yield and loading surfaces we represent the stress–strain response in a two-dimensional stress-plastic strain space and introduce the concept of the upper and lower quasistatic stress–strain curve for a given temperature. Finally, the location of the thermodynamic reference stress for various amounts of plastic deformation is shown to be related to the family of quasistatic stress–strain curves for different temperatures.

In Section 4 previously obtained experimental results for aluminum[3] are interpreted in terms of theory presented in Sections 2 and 3 and conclusions are drawn concerning (1) the equilibrium stress–strain line, (2) the size of the yield surface as a function of the plastic strain, (3) the temperature at which the width of the yield surface becomes zero and its relation with the plastic strain, (4) the length of the straight line to which the yield surface degenerates at the

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‖Earlier, Green and Naghdi[5] developed the theory in the specific context of elastic–plastic materials.

above maximum temperature and its relation with the plastic strain, and (5) the relation between the magnitude of the thermodynamic reference stress and the plastic strain.

## 2. GENERAL THEORY

We assume that for every temperature $\theta = \theta^*$, there exists a region in stress space defined by scalar functions $f$ and $\kappa$

$$f(\sigma_{kl}, \theta^*, \text{history of deformation}) \leqq \kappa(\theta^*, \text{history of deformation}) \tag{1}$$

for which the *incremental* mechanical response is governed by a linear relation in stress $\sigma_{kl}$, strain $\epsilon_{kl}$, and temperature. In stress-temperature configuration space, there exists, at each instant in the history of the material a domain of elastic response. The boundary of this domain is defined by

$$f(\sigma_{kl}, \theta, \text{history of deformation}) = \kappa(\theta, \text{history of deformation}). \tag{2}$$

The one parameter family of surfaces ($\theta$ = constant) represents the yield surfaces at specified temperatures. The size of the elastic region bounded by the yield surfaces is assumed to be a monotonically non-increasing function of thermodynamic temperature $\theta$.

It is assumed that the elastic domain is unaltered by purely elastic deformations, i.e. by stress histories $\sigma_{ij}(t)$ such that inequality (1) is satisfied for all times $t$ in the interval $[t_a, t_b]$. Thus, if for all times $t$ in the interval $[t_a, t_b)$

$$f < \kappa$$

or

$$f = \kappa,\ \frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial f}{\partial \theta}\dot{\theta} \leqq 0 \tag{3}$$

where $(\cdot)$ denotes differentiation with respect to time,† a purely elastic process is said to occur.

Let the strain be written as the sum of two components

$$\epsilon_{kl} = \epsilon'_{kl} + \epsilon''_{kl} \tag{4}$$

such that $\dot{\epsilon}''_{kl} = 0$ whenever (3) is satisfied and $\dot{\epsilon}'_{kl}$ is a linear function of $\dot{\sigma}_{kl}$ and $\dot{\theta}$ under the same conditions. It should be noted that for the present we decline to characterize $\epsilon'_{kl}$ and $\epsilon''_{kl}$ by the adjectives, elastic and plastic, respectively. Nonetheless, the quantities $\epsilon'_{kl}$ and $\epsilon''_{kl}$ as defined above possess many of the attributes suggested by such terminology.

We now assume that there exist functions

$$\psi = \psi(\epsilon'_{kl}, \epsilon''_{kl}, k^{(i)}, \theta) \tag{5}$$

$$s = s(\epsilon'_{kl}, \epsilon''_{kl}, k^{(i)}, \theta) \tag{6}$$

$$q_k = q_k(\epsilon'_{kl}, \epsilon''_{kl}, k^{(i)}, \theta, \theta_{,k}) \tag{7}$$

where $\psi$ is the specific Helmholtz free energy, $s$ is the specific entropy, $q_k$ is the outward heat flux vector, and $\theta_{,k}$ is the spatial temperature gradient. The parameters $k^{(i)}$, $i = 1, 2, \ldots$ represent a finite number of measures of the effects of prior deformation history upon the material response.

Since the elastic domain has been assumed to be invariant whenever (3) is satisfied, the yield condition (2) must be independent of $\epsilon'_{kl}$ under these circumstances.

Except for eqn (4) we have made no postulates as to the interpretation of, or properties of $\epsilon'_{kl}$ and $\epsilon''_{kl}$ when condition (3) is not satisfied. Thus, without loss in generality, we can assume for simplicity that the history dependence of $f$ and $\kappa$ is independent of $\epsilon'_{kl}(\tau)$, $-\infty < \tau \leqq t$. We

†At $t = t_b$ we need only require that $f \leqq \kappa$. The absence of a term which reflects the dependence of $f$ on history in (3) follows from the assumed invariance of the elastic domain under purely elastic deformations.

now postulate that all of the parameters $k^{(i)}$ necessary to describe the material response are governed by evolution-type constitutive relations of the form†

$$\dot{k}^{(i)} = g^{(i)}(\sigma_{kl}, \epsilon''_{kl}, k^{(i)}, \theta, \dot{\epsilon}''_{kl}) \tag{8}$$

where the $g(i)$ are assumed to be homogeneous of degree one in $\dot{\epsilon}''_{kl}$. Thus, the functions $f$ and $\kappa$ may be written in the more explicit forms

$$f = f(\sigma_{kl}, \theta, \epsilon''_{kl}, k^{(i)}) \tag{9}$$

$$\kappa = \kappa(\theta, \epsilon''_{kl}, k^{(i)}). \tag{10}$$

The total rate of change of $f$ is

$$\dot{f} = \frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial f}{\partial \theta}\dot{\theta} + \frac{\partial f}{\partial \epsilon''_{kl}}\dot{\epsilon}''_{kl} + \sum_{i=1}^{N}\frac{\partial f}{\partial k^{(i)}}\dot{k}^{(i)}. \tag{11}$$

where the summation convention has been employed on the tensor subscripts. Equation (11) may be written in the more compact form

$$\dot{f} = \frac{\partial f}{\partial \sigma_{kl}}\dot{\sigma}_{kl} + \frac{\partial f}{\partial \theta}\dot{\theta} + \frac{\delta f}{\delta \epsilon''_{kl}}\dot{\epsilon}''_{kl} \tag{12}$$

by introducing the operator

$$\frac{\delta}{\delta \epsilon''_{kl}} \equiv \frac{\partial}{\partial \epsilon''_{kl}} + \sum_{i=1}^{N}\frac{\partial}{\partial k^{(i)}}\frac{\partial g^{(i)}}{\partial \dot{\epsilon}''_{kl}}. \tag{13}$$

We proceed with the thermodynamic analysis of our material. From the Clausius–Duhem inequality we conclude that

$$s = -\frac{\partial \psi}{\partial \theta} \tag{14}$$

$$\sigma_{kl} = \rho\frac{\partial \psi}{\partial \epsilon'_{kl}} \tag{15}$$

when the material is in an elastic state. In eqn (15) $\rho$ is the mass density.‡ We postulate that eqns (14) and (15) apply during plastic deformation processes as well; and by so doing, define $\epsilon'_{kl}$ for all deformation processes.

From eqn (15) we conclude that during any elastic process

$$\mathrm{d}\sigma_{kl} = \rho\frac{\partial^2 \psi}{\partial \epsilon'_{kl}\partial \epsilon'_{pq}}\mathrm{d}\epsilon'_{pq} + \rho\frac{\partial^2 \psi}{\partial \epsilon'_{kl}\partial \theta}\mathrm{d}\theta. \tag{16}$$

If the theory is to predict that the elastic incremental stress–strain-temperature relation is to be insensitive to the history of inelastic deformation[12] then

$$\frac{\partial^2 \psi}{\partial \epsilon'_{kl}\partial \epsilon'_{pq}} = f^{(1)}_{klpq}(\epsilon'_{mn}, \theta) \tag{17}$$

$$\frac{\partial^2 \psi}{\partial \epsilon'_{kl}\partial \theta} = f^{(2)}_{kl}(\epsilon'_{mn}, \theta). \tag{18}$$

†Kratochvil and Dillon[6] similarly postulate a linear dependence of $\dot{k}^{(i)}$ on $\dot{\epsilon}''_{kl}$ and assert that this assumption valid for materials in which thermally activated mechanisms of dislocation motion may be neglected. Kröner and Teodosiu[7] discussed the conditions under which this assumption is valid.

‡The theory can be extended to finite deformation by identifying $\epsilon$, $\sigma$ and ρ with the Green strain tensor, the second Piola–Kirchoff stress tensor, and the density in the reference configuration. The applicability of the additive decomposition of strain and its relation to Lee's[8] decomposition has been discussed by Green and Naghdi[9] and more recently by Sidoroff[10] and Kleiber[11]. Since we shall postulate that linear incremental elastic relations are unaffected by inelastic deformation, gross distortions are ruled out. For our present purposes, it suffices to consider linear kinematics of infinitesimal deformation.

From eqn (17)

$$\frac{\partial \psi}{\partial \epsilon'_{kl}} = f^{(3)}_{kl}(\epsilon'_{mn}, \theta) + f^{(4)}_{kl}(\epsilon''_{mn}, k^{(i)}, \theta) \tag{19}$$

and (18) and (19) then require that

$$f^{(2)}_{kl}(\epsilon'_{mn}, \theta) = \frac{\partial f^{(3)}_{kl}}{\partial \theta}(\epsilon'_{mn}, \theta) + \frac{\partial f^{(4)}_{kl}}{\partial \theta}(\epsilon''_{mn}, k^{(i)}, \theta). \tag{20}$$

From

$$f^{(2)}_{kl}(\epsilon'_{mn}, \theta) - \frac{\partial f^{(3)}_{kl}(\epsilon'_{mn}, \theta)}{\partial \theta} \equiv f^{(5)}_{kl}(\epsilon'_{mn}, \theta) = \frac{\partial f^{(4)}_{kl}(\epsilon''_{mn}, k^{(i)}, \theta)}{\partial \theta} \tag{21}$$

we conclude that $f^{(5)}_{kl}$ and $\partial f^{(4)}_{kl}/\partial \theta$ may depend, at most, on $\theta$, and that

$$f^{(5)}_{kl}(\theta) = \frac{\partial f^{(4)}_{kl}}{\partial \theta}$$

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$$f^{(4)}_{kl} = f^{(6)}_{kl}(\theta) - \phi_{kl}(\epsilon''_{mn}, k^{(i)}). \tag{22}$$

By substituting (22) into (19) and integrating, we obtain

$$\psi = \psi'(\epsilon'_{mn}, \theta) - \epsilon'_{kl}\phi_{kl}(\epsilon''_{mn}, k^{(i)}) + f^{(7)}(\epsilon''_{mn}, k^{(i)}, \theta). \tag{23}$$

From eqns (14) and (23) we conclude that during any elastic process

$$\mathrm{d}s = -\frac{\partial^2 \psi'}{\partial \epsilon'_{kl}\partial \theta}\mathrm{d}\epsilon'_{kl} - \frac{\partial^2 \psi'}{\partial \theta^2}\mathrm{d}\theta - \frac{\partial^2 f^{(7)}}{\partial \theta^2}\mathrm{d}\theta. \tag{24}$$

If the theory is to predict that the elastic incremental entropy-strain-temperature relation is to be insensitive to the history of inelastic deformation, then

$$\frac{\partial^2 f^{(7)}}{\partial \theta^2} = f^{(8)}(\theta). \tag{25}$$

Integrating eqn (25) we obtain

$$f^{(7)}(\epsilon''_{mn}, k^{(i)}, \theta) = f^{(9)}(\theta) - \theta \cdot h^{(1)}(\epsilon''_{mn}, k^{(i)}) + h^{(2)}(\epsilon''_{mn}, k^{(i)}) \tag{25a}$$

which includes the two functions $h^{(1)}(\epsilon''_{mn}, k^{(i)})$ and $h^{(2)}(\epsilon''_{mn}, k^{(i)})$ still to be interpreted. We now introduce for convenience the function $\theta_a(\epsilon''_{mn}, k^{(i)})$ into eqn (25a) and obtain

$$\begin{aligned} f^{(7)}(\epsilon''_{mn}, k^{(i)}, \theta) = f^{(9)}(\theta) - (\theta - \theta_a(\epsilon''_{mn}, k^{(i)}))h^{(1)}(\epsilon''_{mn}, k^{(i)}) \\ + \psi''(\epsilon''_{mn}, k^{(i)}) \end{aligned} \tag{26}$$

where

$$\psi''(\epsilon''_{mn}, k^{(i)}) = h^{(2)}(\epsilon''_{mn}, k^{(i)}) - \theta_a(\epsilon''_{mn}, k^{(i)})h^{(1)}(\epsilon''_{mn}, k^{(i)}).$$

From eqns (23) and (26) we obtain†

$$\begin{aligned} \psi = \psi'(\epsilon'_{mn}, \theta) + \psi''(\epsilon''_{mn}, k^{(i)}) - \epsilon'_{kl}\phi_{kl}(\epsilon''_{mn}, k^{(i)}) \\ - (\theta - \theta_a(\epsilon''_{mn}, k^{(i)}))s''(\epsilon''_{mn}, k^{(i)}) \end{aligned} \tag{27}$$

†Lubliner[13] and Naghdi and Trapp[14] derive relations to which (27) reduces if we neglect $\phi_{kl}$ and $s''$, respectively. Lubliner identifies $s''$ with the "configurational entropy" discussed by Cottrell[15].

where $f^{(9)}$ was incorporated into $\psi'$ and we have replaced $h^{(1)}$ by the more descriptive notation $s''$. From eqns (14) and (27)

$$s = -\frac{\partial \psi'(\epsilon'_{mn}, \theta)}{\partial \theta} + s''(\epsilon''_{mn}, k^{(i)}). \tag{28}$$

Similarly eqns (15) and (27) imply that

$$\sigma_{kl} = \rho\left[\frac{\partial \psi'(\epsilon'_{mn}, \theta)}{\partial \epsilon'_{kl}} - \phi_{kl}(\epsilon''_{mn}, k^{(i)})\right]. \tag{29}$$

The incremental relations for entropy and stress become

$$\mathrm{d}s = -\frac{\partial^2 \psi'}{\partial \theta \partial \epsilon'_{mn}}\mathrm{d}\epsilon'_{mn} - \frac{\partial^2 \psi'}{\partial \theta^2}\mathrm{d}\theta + \frac{\delta s''}{\delta \epsilon''_{mn}}\mathrm{d}\epsilon''_{mn} \tag{30}$$

$$\mathrm{d}\sigma_{kl} = \rho\left[\frac{\partial^2 \psi'}{\partial \epsilon'_{kl}\partial \epsilon'_{mn}}\mathrm{d}\epsilon'_{mn} + \frac{\partial^2 \psi'}{\partial \epsilon'_{kl}\partial \theta}\mathrm{d}\theta - \frac{\delta \phi_{kl}}{\delta \epsilon''_{mn}}\mathrm{d}\epsilon''_{mn}\right] \tag{31}$$

which reduce to the incremental elastic relations whenever inequality (3) is satisfied.

In addition to eqns (14) and (15) it can be readily demonstrated that whenever inequality (3) is satisfied,

$$-q_k\theta_{,k} \geqq 0 \tag{32}$$

holds; and for homogeneous temperature distribution†

$$(\sigma_{kl} - \sigma^0_{kl})\dot{\epsilon}''_{kl} \geqq 0 \tag{33}$$

where the "thermodynamic reference stress"[1] is defined as

$$\sigma^0_{kl} \equiv \rho\frac{\delta \psi}{\delta \epsilon''_{kl}}. \tag{34}$$

From (27) and (34)

$$\sigma^0_{kl} = \rho\left[\frac{\delta \psi''}{\delta \epsilon''_{kl}} - \epsilon'_{mn}\frac{\delta \phi_{mn}}{\delta \epsilon''_{kl}} - (\theta - \theta_a)\frac{\delta s''}{\delta \epsilon''_{kl}} + \frac{\delta \theta_a}{\delta \epsilon''_{kl}}s''\right]. \tag{35}$$

Alternatively, we can write

$$\sigma^0_{kl} = \rho\left[\frac{\delta \psi^{(2)}}{\delta \epsilon''_{kl}} - \epsilon'_{mn}\frac{\delta \phi_{mn}}{\delta \epsilon''_{kl}} - \theta\frac{\delta s''}{\delta \epsilon''_{kl}}\right] \tag{36}$$

where

$$\psi^{(2)} \equiv \psi'' + \theta_a s''. \tag{37}$$

In [1] it was concluded from (33) that $\sigma^0_{kl}$ must lie within or on the current yield surface and so an additional restriction must be satisfied by the constitutive equations. The basis for this conclusion must now be reconsidered. In [1] it was also assumed that the $\sigma_{ij}$ must always lie within or on the yield surface; whereas it has subsequently been demonstrated[3] that, in general, $\sigma_{ij}$ lies outside of the yield surface during a typical inelastic event. If the loading is terminated and $\sigma_{ij}$ held constant, additional inelastic or creep strains will develop. With the development of such creep strains the yield surface gradually approaches the now stationary stress point $\sigma_{ij}$. We are thus led to the concept of an equilibrium or steady state yield surface. It follows immediately from the arguments in [16] that *if* $\sigma^0_{kl}$ is independent of $\epsilon'_{kl}$ and (a) the

†It can be shown[4] that (32) and (33) hold for all processes if $q_k$ is independent of $\epsilon''_{kl}$ and $k^{(i)}$.

loading is done at a sufficiently slow rate that $f=\kappa$; or (b) the stress is held constant so that $f$ approaches $\kappa$ with time; or (c) the stress is reduced sufficiently so that the $f\leqq\kappa$; *then*, at the instant of resumption of loading, it is *necessary* that $\sigma^0_{kl}$ lie within or on the current yield surface. It can be shown that, in the absence of the above restrictions on the location of the current stress point and upon the independence of $\sigma^0_{kl}$ on $\epsilon'_{kl}$, the confinement of $\sigma^0_{kl}$ to within or on the current yield surface is *sufficient* to permit satisfaction of the Clausius–Duhem inequality. We shall examine the *necessary* condition on the location of $\sigma^0_{kl}$ for the important case when the first restriction is enforced, i.e. when (a) or (b) or (c) prevails. But first it shall be necessary to consider the nature of the constitutive relation which governs $\epsilon'_{kl}$ in some detail.

If we assume that Hooke's law with temperature-dependent moduli governs the elastic relations when inequality (3) holds, then from (31)

$$\rho\frac{\partial^2\psi'}{\partial\epsilon'_{kl}\partial\epsilon'_{mn}}=K_{klm}(\theta)=K_{mnkl}(\theta) \tag{38}$$

$$\rho\frac{\partial^2\psi'}{\partial\epsilon'_{kl}\partial\theta}=\beta_{kl}(\theta). \tag{39}$$

If we define an elastic compliance $K^{-1}_{klmn}$, such that

$$K^{-1}_{klmn}K_{klpq}=\delta_{mp}\delta_{nq} \tag{40}$$

then (31) may be inverted to produce

$$d\epsilon'_{pq}=K^{-1}_{klpq}\,d\sigma_{kl}-K^{-1}_{klpq}\beta_{kl}\,d\theta+\rho K^{-1}_{klpq}\frac{\delta\phi_{kl}}{\delta\epsilon''_{mn}}\,d\epsilon''_{mn}. \tag{41}$$

Equation (41) leads one to the somewhat surprising conclusion that whenever unloading occurs $d\epsilon'_{pq}$ obeys the incremental elastic constitutive relation of the virgin material; yet during loading the elastic relation is not valid unless $\phi_{kl}$ vanishes.† Thus, in some respects our decomposition of the strain tensor (4) does not lead to quantities which satisfy completely the intuitive connotations of the terms elastic and plastic. Alternatively we can write

$$d\epsilon_{ij}=d\epsilon^e_{ij}+d\epsilon^p_{ij} \tag{42}$$

where

$$d\epsilon^e_{ij}=K^{-1}_{klij}\,d\sigma_{kl}-K^{-1}_{klij}\beta_{kl}\,d\theta \tag{43}$$

$$d\epsilon^p_{ij}=\left[\delta_{ip}\delta_{jq}+\rho K^{-1}_{klij}\frac{\delta\phi_{kl}}{\delta\epsilon''_{pq}}\right]d\epsilon''_{pq} \tag{44}$$

and $\delta_{ij}$ is the Kronecker delta. It follows that

$$\epsilon^p_{ij}=\epsilon''_{ij}+\rho K^{-1}_{klij}\phi_{kl} \tag{45}$$

assuming that $\epsilon^p_{ij}=\epsilon''_{ij}=\phi_{kl}=0$ for the virgin material, and that $K^{-1}_{klij}$ is constant over the range of integrations. Under the additional assumption of constant coefficients of thermal expansion, integration of (43) results in

$$\epsilon^e_{ij}=K^{-1}_{klij}\sigma_{kl}-K^{-1}_{klij}\beta_{kl}\theta=\epsilon'_{ij}-\rho K^{-1}_{klij}\phi_{kl}. \tag{46}$$

We shall call $\epsilon^p_{ij}$ and $\epsilon^e_{ij}$ the elastic and plastic strain components.‡

†A constant $\phi_{kl}$ could be absorbed into the function $\psi^{(1)}$ in eqn (27) and thus $\phi_{kl}=$ const. has no significance.

‡The quantities $\epsilon'_{ij}$ and $\epsilon^e_{ij}$ represent hypoelastic and hyperelastic definitions of elastic strain components since, in the inelastic domain, they preserve the elasticity relationship through the Helmholtz potential, and through the incremental Hooke's law, respectively.

We may now return to the consideration of the restriction placed by inequality (33) upon the thermodynamic reference stress and hence upon the constitutive relations.

Phillips and his coworkers[3, 17, 18] have shown that at a sufficiently high temperature the region of elastic response becomes vanishingly small. Figure 1 shows a conceptualization of the family of yield surfaces which bound regions of elastic response for various temperatures but a common prior history of plastic deformation. It is assumed that the envelope in stress-temperature space has a uniquely defined† apex $\sigma_{ij}^*$ at a temperature $\theta_a$.

In the limit as $\theta$ approaches $\theta_a$ the permissible values of stress are restricted to an infinitesimal neighborhood of $\sigma_{ij}^*$. Thus $\epsilon'_{ij}$ also approaches a limit $\epsilon'^*_{ij}$ and so for any specified prior history of plastic deformation $\sigma_{kl}^0$ approaches a fixed limit. Unless the high temperature limit of $\sigma_{kl}^0$ is identical with $\sigma_{kl}^*$, the high temperature limit of the elastic region, inequality (33) will be violated for some loading $d\sigma$. From eqn (35) we conclude that

$$\sigma_{kl}^0 = \sigma_{kl}^* - (\theta - \theta_a)c_{kl} - d_{klmn}(\epsilon'_{mn} - \epsilon'^*_{mn}) \tag{47}$$

where

$$c_{kl} = \rho \cdot \delta s''/\delta\epsilon''_{kl} \tag{48}$$

$$d_{klmn} = \rho \cdot \delta\phi_{mn}/\delta\epsilon''_{kl} \tag{49}$$

$$\sigma_{kl}^* = \rho\left[\frac{\delta\psi''}{\delta\epsilon''_{kl}} - \epsilon'^*_{mn}\frac{\delta\phi_{mn}}{\delta\epsilon''_{kl}} + \frac{\delta\theta_a}{\delta\epsilon''_{kl}}s''\right] \tag{50}$$

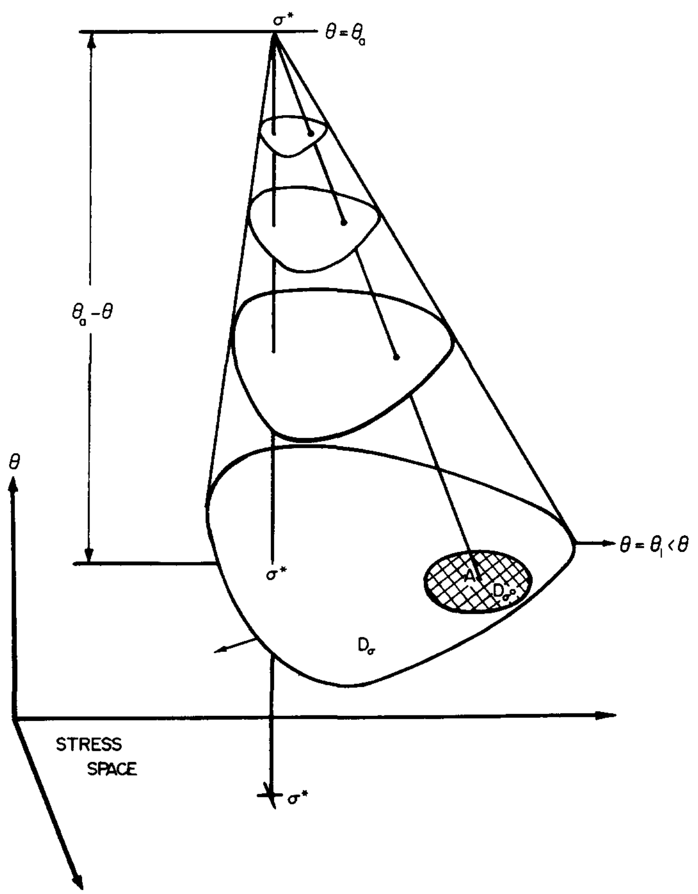


Fig. 1. Family of yield surfaces.

†It is possible, indeed probable, that some degree of indeterminacy exists. For example, if hydrostatic stress has no influence on yielding at elevated temperatures near $\theta_a$ then the limit $\sigma_{ij}^*$ will have an indeterminate hydrostatic component. Additional discussion of the uniqueness of $\sigma_{ij}^*$ is deferred to the next section.

depend only on the history of the inelastic deformation which gave rise to the family of yield surfaces shown in Fig. 1. In earlier theories it was assumed that $c_{kl} = d_{klmn} = 0$ so that $\sigma^0_{kl}$ and $\sigma^*_{kl}$ are identical. The second term in (47) causes a linear departure of $\sigma^0_{kl}$ from $\sigma^*_{kl}$ with decreasing temperature. If $d_{klmn} = 0$, $\sigma^0_{kl}$ is independent of $\epsilon'_{mn}$ and so $\sigma^0_{kl}$ must lie within or on the yield surface if (33) is to be satisfied. This restricts the gradient $c_{kl}$. If $d_{klmn} \neq 0$ the conditions under which (33) is satisfied require careful analysis.

Let $\sigma_{kl}$ be contained within the domain of elastic response $D_\sigma$ appropriate to the temperature $\theta_1 < \theta_a$. From (46) we conclude that $\epsilon'_{ij}$ occupies a domain $D_\epsilon$ in strain space which is a one-to-one mapping of $D_\sigma$. From (47) and (49) we conclude that $\sigma^0_{kl}$ occupies a domain $D_{\sigma^0}$ (indicated by the crosshatched region in Fig. 1) in the neighborhood of the point $A$ calculated from the first two terms of (47). A *sufficient* condition for the satisfaction of (33) is that $D_{\sigma^0}$ be a subdomain of $D_\sigma$. If $d_{klmn} \neq 0$ it is possible for $\sigma^0_{kl}$ to lie outside the yield surface. As $\sigma_{ij}$ traverses the yield surface or boundary of $D_\sigma$, $\sigma^0_{ij}$ traverses the boundary of $D_{\sigma^0}$. To each point $\sigma_{ij}(A, B, C, \ldots)$ there corresponds a point $\sigma^0_{ij}(A', B', C', \ldots)$ as shown in Fig. 2. When $\sigma_{ij}$ is at $A$, inequality (33) requires that $\sigma^0_{ij}$ lie behind the hyperplane tangent to the yield surface at $A$. Similarly for $B, C, \ldots$ If $\sigma^0_{ij}$ is stationary as $\sigma_{ij}$ traverses the yield surface then $\sigma^0_{ij}$ must lie inside the yield surface. If $\sigma^0_{ij}$ is a function of $\epsilon'_{ij}$ and therefore non-stationary it is possible to satisfy (33) in the manner shown in Fig. 2.

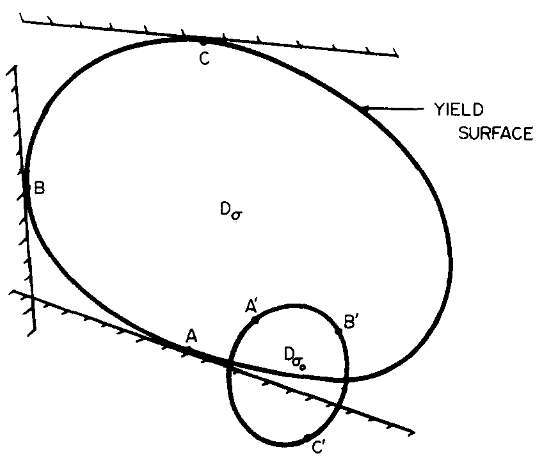


Fig. 2. The domains $D_\sigma$ and $D_{\sigma^0}$.

## 3. SIMPLIFIED THEORY

For our current purposes we shall *assume* that $\phi_{kl} = 0$. Under this hypothesis, the alternative strain decompositions (4) and (42) become congruent and $\sigma^0_{ij}$ becomes independent of $\epsilon'_{ij}$. If we also *assume* $s'' = 0$ then $\sigma^*_{ij} = \sigma^0_{ij}$. Thus, in the simplified theory it is assumed that the thermodynamic reference stress $\sigma^0_{ij}$ has the value of its high temperature limit $\sigma^*_{ij}$ for all temperatures. In adopting the hypotheses $\phi_{kl} = 0$, $s'' = 0$ we reduce our theory to the more classical form but leave as an open question the experimental means by which these quantities may be quantitatively evaluated and hence the conditions under which the retention of $\phi_{kl}$ and $s''$ in the more general theory may be required.

We shall further restrict our considerations to the analysis of monotonically increasing radial loadings. With the guidance of some experimental observations of Phillips and Tang[3] we shall show that the above hypotheses lead to a theory which is in qualitative agreement with experiment.

It has been demonstrated[19, 20] that there exists both a yield surface and a loading surface in stress space. The yield surface bounds a domain of purely elastic response and, for a class of quasistatic loading paths which include monotonic radial loading, the yield surface is tangent to the loading surface. The loading surface which appears, to a first approximation, to be close to the Mises surface encloses a region of stress space in which the rate of generation of plastic strain per unit loading is small. Moreover, it is observed that for quasistatic loading the plastic strain rate vector is normal to both the yield surface and loading surface. Figure 3 shows the qualitative features of this model for very slow (quasistatic) monotonic radial loading in combined tension and torsion. For now, the only use that we make of these observations is to conclude that, for such monotonic loading it is reasonable to define the magnitude of the stress vector by means of a norm which is related to the classical Mises criterion.

Let the square of the differential length in stress space be defined by

$$(\mathrm{d}l)^2 = G_{klmn}\,\mathrm{d}\sigma_{kl}\,\mathrm{d}\sigma_{mn} \tag{51}$$

where

$$G_{klmn} = \frac{3}{2}\left[\delta_{mk}\delta_{nl} - \frac{1}{3}\delta_{kl}\delta_{mn}\right]. \tag{52}$$

We can define a new set of variables $\bar{\sigma}_{ij}$ which are proportional to the deviatoric stress by the linear transformation

$$\sigma_{kl} = \sqrt{\frac{2}{3}}\left[\bar{\sigma}_{kl} + \frac{1}{3}\delta_{kl}\sigma_{pp}\right]. \tag{53}$$

From eqns (51)–(53) we find

$$(\mathrm{d}l)^2 = \mathrm{d}\bar{\sigma}_{kl}\,\mathrm{d}\bar{\sigma}_{kl} = \delta_{km}\delta_{ln}\,\mathrm{d}\bar{\sigma}_{kl}\,\mathrm{d}\bar{\sigma}_{mn}. \tag{54}$$

At first glance, it appears that our choice of the norm would define the space to be a nine-dimensional Riemannian manifold $R_9$ reducible to a Euclidean manifold $E_9$ in which the components of $\bar{\sigma}_{ij}$ play the role of a cartesian reference frame. This is not quite the case, since the tensor $G_{klmn}$ is not a positive definite metric.† Consequently we can formally compute magnitude by means of the Euclidean norm in terms of $\bar{\sigma}_{ij}$ but must recognize that the transformation (53) is not linearly independent and so not all points in the space are realizable. If we assume symmetry then we can define a five-dimensional reference frame $\bar{\sigma}_i$ such that

$$\mathrm{d}l^2 = 2[(\mathrm{d}\bar{\sigma}_1)^2 + (\mathrm{d}\bar{\sigma}_2)^2 + (\mathrm{d}\bar{\sigma}_3)^2 + (\mathrm{d}\bar{\sigma}_4)^2 + (\mathrm{d}\bar{\sigma}_5)^2 + (\mathrm{d}\bar{\sigma}_1)(\mathrm{d}\bar{\sigma}_2)] \tag{55}$$

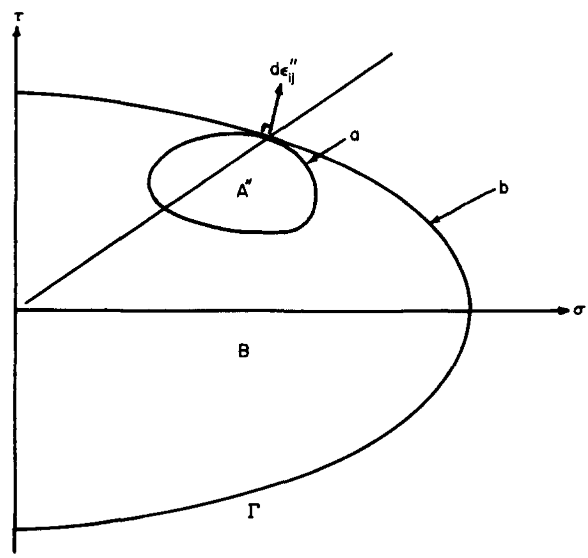


Fig. 3. The yield surface and the loading surface. (a) Yield surface, (b) loading surface, $A$: elastic region; $B$: region of small plastic strain rate; Γ: region of large plastic strain rate.

†It is positive semidefinite since it defines the distance between any two stress points which differ by a hydrostatic component to be zero. In effect, $G_{klmn}$ reduces the principal stress space to the $\pi$-plane.

where

$$\bar{\sigma}_1 = \bar{\sigma}_{11}, \quad \bar{\sigma}_2 = \bar{\sigma}_{22}, \quad \bar{\sigma}_3 = \bar{\sigma}_{12}, \quad \bar{\sigma}_4 = \bar{\sigma}_{23}, \quad \bar{\sigma}_5 = \bar{\sigma}_{31}.$$

It can be shown that there exists a simple transformation $\bar{\tau}_i = c_{ij}\bar{\sigma}_j$ which will produce a Euclidean metric $(\mathrm{d}l)^2 = (\mathrm{d}\bar{\tau}_i)(\mathrm{d}\bar{\tau}_i)$. Thus, the stress space is equivalent to a Euclidean space $E_5$, but we shall find it more convenient to operate in terms of eqn (54).

We define

$$\bar{\epsilon}^p_{kl} = \sqrt{\frac{2}{3}}\left(\epsilon^p_{kl} - \frac{1}{3}\delta_{kl}\rho^p_{pp}\right) \tag{56}$$

and assume plastic incompressibility $\epsilon^p_{pp} = 0$, so inequality (33) becomes

$$[\bar{\sigma}_{kl} - \bar{\sigma}^0_{kl}]\dot{\bar{\epsilon}}^p_{kl} \geqq 0. \tag{57}$$

Since for the monotonic radial loading, both $\bar{\sigma}_{kl}$ and $\dot{\bar{\epsilon}}_{kl}$ are normal to the Mises loading surface we conclude that

$$\bar{\sigma}_{kl} = \bar{\sigma}\lambda_{kl}, \quad \dot{\bar{\epsilon}}^p_{kl} = \dot{\bar{\epsilon}}^p\lambda_{kl}$$

where $\lambda_{kl}$ is a unit tensor in the $\bar{\sigma}_{kl}$ cartesian reference frame. From (54) we conclude

$$\lambda_{kl}\lambda_{kl} = 1 \tag{58}$$

so, from (57)

$$(\bar{\sigma} - a\bar{\sigma}^0)\dot{\bar{\epsilon}}^p \geqq 0 \tag{59}$$

where $\bar{\sigma}^0_{kl} = \mu_{kl}\bar{\sigma}^0$, $a = \mu_{kl}\lambda_{kl}$, and $\bar{\sigma}$, $\bar{\sigma}^0$, and $\dot{\bar{\epsilon}}^p$ represent the magnitudes of the tensors in (57). Since the magnitudes are intrinsically positive

$$\bar{\sigma} \geqq a\bar{\sigma}^0 \tag{60}$$

where

$$a = \cos \sphericalangle(\dot{\bar{\epsilon}}^p_{kl}, \bar{\sigma}^0_{kl}) \leqq 1.$$

From (35) we conclude that

$$\sigma^0_{kl} = \rho\frac{\mathrm{d}\psi''}{\mathrm{d}\epsilon^p_{kl}} = \rho\frac{\mathrm{d}\psi''}{\mathrm{d}\bar{\epsilon}^p_{kl}} = \rho\frac{\mathrm{d}\psi''}{\mathrm{d}\bar{\epsilon}^p}\frac{\mathrm{d}\bar{\epsilon}^p}{\mathrm{d}\bar{\epsilon}^p_{kl}} \tag{61}$$

where we have used the total differential symbol since, for monotonically increasing radial plastic strain trajectories, the work-hardening parameters defined by the evolution eqn (8) are uniquely related to $\epsilon^p_{kl}$.

From (58) and (61) we conclude

$$\sigma^0_{kl} = \bar{\sigma}^0_{kl} = \rho\frac{\partial\psi}{\partial\bar{\epsilon}^p}\lambda_{kl}. \tag{62}$$

Thus, the magnitude of $\sigma^0_{kl}$ is $\bar{\sigma}^0 = \rho(\partial\psi/\partial\bar{\epsilon}^p)$ and its direction is parallel to $\dot{\bar{\epsilon}}_{kl}$. Thus (60) reduces to $\bar{\sigma} \geqq \bar{\sigma}^0$ as expected.

The conclusion that $\bar{\sigma}^0_{kl}$ translates in the direction of $\dot{\bar{\epsilon}}^p_{kl}$ and hence in the direction $\bar{\sigma}_{kl}$ for the radial loading case is of some interest. Phillips and his co-workers[3, 17] have shown that under quite general conditions the yield surface translates in the direction of pre-stressing. Since the size of the elastic region is often quite limited and since $\sigma^0_{kl}$ must lie within the current yield surface (under the restrictive assumptions of the simplified theory), the theoretical conclusion that a special interior point $\sigma^0_{kl}$ moves in the experimentally observed direction of the overall yield surface is reassuring evidence of a degree of internal consistency in the theory.

By introducing the stress magnitude $\bar{\sigma}$ we can simplify the description of the yield surfaces in stress-temperature space. The qualitative features of the behavior can be visualized by projecting the three-dimensional representation of Fig. 1 into a $\bar{\sigma}-\theta$ plane.

The stress–strain response can now be represented in a plane of $\bar{\sigma}$ vs $\bar{\epsilon}^p$. In such a plane we assume that for all radial loadings at the same stress rate $\dot{\bar{\sigma}}$ and the same temperature $\theta$ there is a unique curve $AB$, Fig. 4, which represents the relationship $\bar{\sigma}=\bar{\sigma}(\bar{\epsilon}^p)$ for the temperature $\theta$ and stress rate $\dot{\bar{\sigma}}$. As in [20] we introduce a quasistatic stress strain curve $AC$, which, for a given temperature $\theta$, is the stress–strain curve corresponding to $\dot{\bar{\sigma}}=0$. It is the sequence of equilibrium positions due to successively larger values of applied stress; that is, each increment of stress is applied only after the permanent strains due to the previous stress increments have developed fully. The significance of the quasistatic stress–strain curve becomes apparent from the following discussion.

If, while obtaining the curve $AB$, we interrupt the increase of stress at $D$ and keep the stress $\bar{\sigma}$ constant, the strain $\bar{\epsilon}^p$ will continue to increase because of creep until the quasistatic curve is reached at $K$ at a final plastic strain $\bar{\epsilon}_K{}^p$. If, however, after the stress reaches the value indicated by $D$, it is decreased to the value indicated by $I$ below the quasistatic line, then the amount of plastic strain $\bar{\epsilon}_E{}^p$ accumulated at the time of crossing at $E$ is frozen. Reloading, we obtain the line $IER$. After the stress during unloading has reached the elastic region below $E$, reloading can occur also in the opposite direction by crossing the limit of the elastic region $\bar{E}$ below $E$. We remark that the prestressing point was at point $D$ in Fig. 4 while the elastic region was at $E\bar{E}$ which is below $D$. Only after the stress remains at $D$ for some time, will creep have raised the elastic region to a level $K\bar{K}$ to pass through the same stress value as the prestress.

It is also possible that the line $AC$ may be so flat that the parallel to $\bar{\epsilon}^p$ from $D$ may never intersect it. Then we have unlimited creep. It can be assumed that the strain rate $\dot{\bar{\epsilon}}^p$ (creep rate) is a monotonically increasing function of the distance $\delta_D$ of the stress level at $D$ from $AC$. Then, as long as this distance decreases we have decreasing strain rate (primary creep); but when this distance remains constant, i.e. when the line $AC$ becomes parallel to $DK$, then we have constant strain rate (secondary creep).

In Fig. 1 it is seen that the size of the elastic domain shrinks with increasing temperature. At some temperature the elastic region vanishes. The temperature at which this occurs will in general depend upon the plastic strain. Thus, in Fig. 4 the equilibrium stress–strain lines $AC$

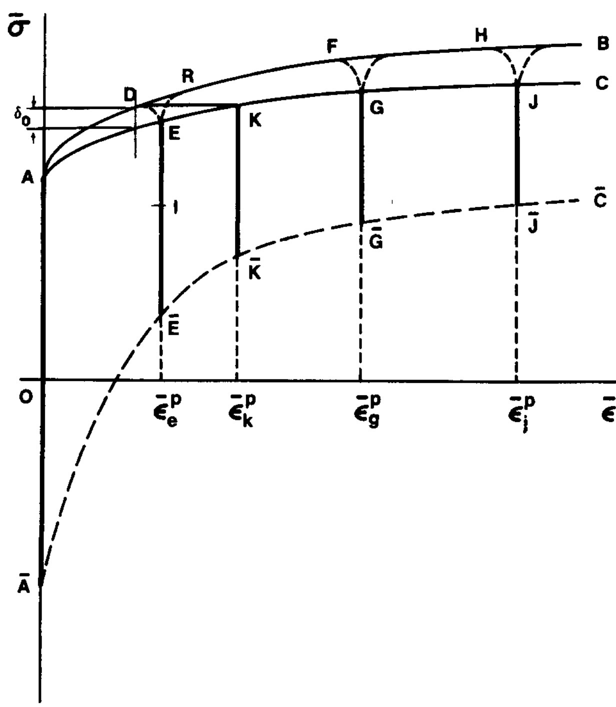


Fig. 4. The $\bar{\sigma}-\bar{\epsilon}^p$ plane and the quasistatic stress–strain curve $AC$.

and $\overline{AC}$ must change positions with the temperature. Figure 5 shows the locus of the intersections of curves $AC$ and $\overline{AC}$ for various values of temperature. Thus, the curve $OP_4P_3PP_2$ also represents the location of the thermodynamic reference stress for various amounts of plastic deformation. Also shown in Fig. 5 is the sequence of equilibrium stress–strain curves for increasing temperature $\theta$. It will be shown in the next section that $\theta_{max}$ decreases in value as $\bar{\sigma}$ increases. Therefore, the locus of intersections of each pair of equilibrium stress–strain curves defines an equilibrium bounding line. Fig. 6 shows a three-dimensional sketch in the $\bar{\sigma}$, $\bar{\epsilon}^p$, $\theta$, space of the curve $\theta_{max} = \theta_{max}(\bar{\sigma}, \bar{\epsilon}^p)$ the projection of which on the $\bar{\sigma} - \bar{\epsilon}^p$ plane gives the equilibrium bounding line of Fig. 5, while its projection on the $\theta - \bar{\sigma}$ plane gives the change in the value of $\theta_{max}$ as prestressing increases.

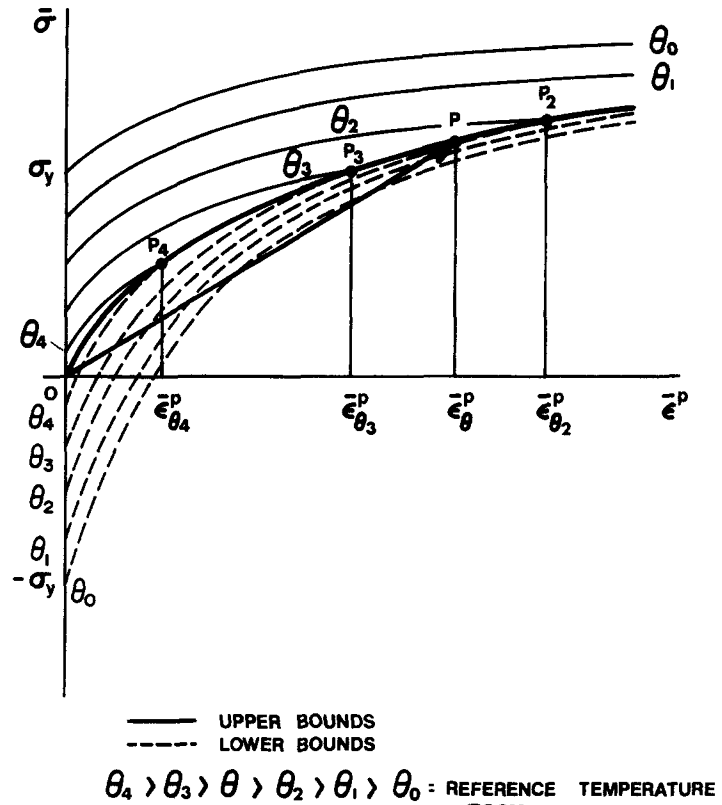


Fig. 5. The quasistatic stress–strain curves at different temperatures and the locus of intersections of each pair of these curves.

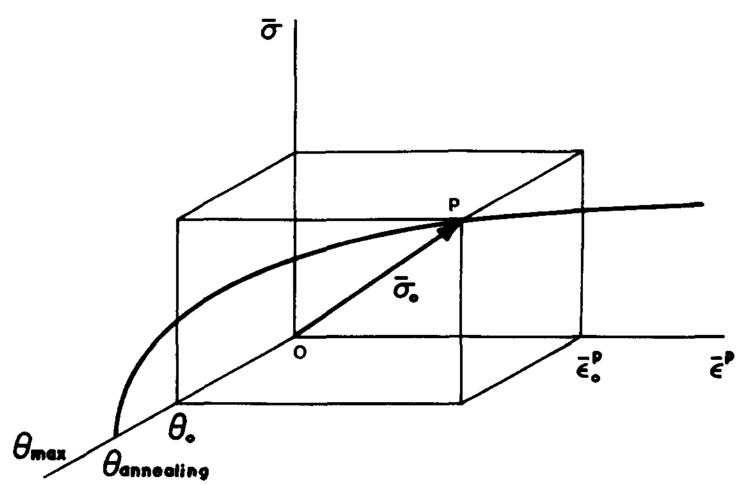


Fig. 6. Three-dimensional representation in the $\bar{\sigma}$, $\bar{\epsilon}^p$, $\theta$, space of the curve $\theta_{max} = \theta_{max}(\bar{\sigma}, \bar{\epsilon}^p)$.

## 4. EXPERIMENTAL RESULTS

In [3, 19] experimental results were presented and on their basis certain conclusions were drawn. These experiments as well as unpublished experiments are labelled $S$-4, $S$-5, . . . ; their schematic loading paths are shown in Fig. 7.† Yield surfaces have been obtained at several prestressing points by obtaining at each point the yield curves corresponding to four different temperatures. Thus the yield surfaces were obtained in the temperature-stress space. For illustration and explanation of the procedure used for obtaining the yield surface we shall use as an example test $S$-10, Fig. 8.

After obtaining the initial yield surface in stress-temperature space on the basis of the four yield curves at temperatures $\theta = 70, 151, 227, 265°$F, the specimen was prestressed to the value $\sigma = 3897$ psi, $\tau = 3897$ psi at temperature $\theta = 70°$F. Upon reaching the prestressing point the stress point was decreased immediately within the subsequent yield surface and this subsequent yield curve was determined. In succession, the yield curves at increasing temperature levels were determined in like fashion. It should be noted that the experimentally determined yield surfaces are zero-offset or proportional limit surfaces obtained by the extrapolation technique described in detail in [3]. We observe that the yield surface does not pass through the prestressing point. This phenomenon was explained in [20] and it was predicted in the previous section. From [19] it can be seen that the points of intersection of a straight line with the four isothermal yield curves can with sufficient accuracy be represented by two straight lines in the stress temperature space. This fact will be used now and we shall draw a series of arbitrary parallel straight lines $A\bar{A}$, $B\bar{B}$, $C\bar{C}$, $D\bar{D}$, $E\bar{E}$ which may be parallel to the prestressing direction, although this is not a requirement. The intersections of these lines with the yield curves are now plotted in a stress-temperature diagram as shown in Fig. 8. We observe that the intersections produce pairs of straight lines $(\bar{A}A', AA')$, $(\bar{B}B', BB')$, . . . . In Fig. 8 we see that the intersections $B'$, $C'$, and $D'$, lie on a straight line $\bar{Q}Q$ which represents a high temperature limit of the sequence of yield curves. This phenomenon was predicted in [2].

Careful examination of the sequence of yield curves at increasing temperatures suggests the validity of an anisotropic sandhill analogy not unlike the classical isotropic sandhill analogy for fully developed ideally plastic torsion [21]. It appears that the slope of the sandhill (maximum $\theta$ gradient) varies with the orientation of the normal to the level surfaces with the preloading

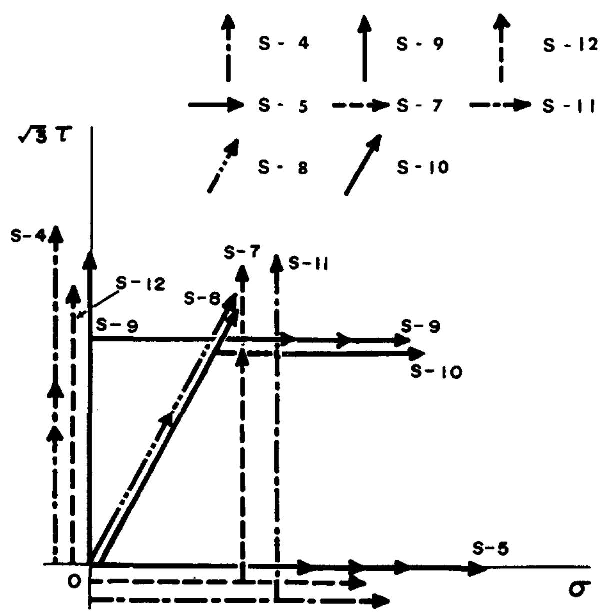


Fig. 7. Stress-paths of experiments.

†Experiments $S$-7, 9, 10, and 11 involve non-radial loading paths and hence the conditions under which the results of the previous section were derived are not satisfied. Nonetheless, they produce results consistent with those of the radial loading path tests.

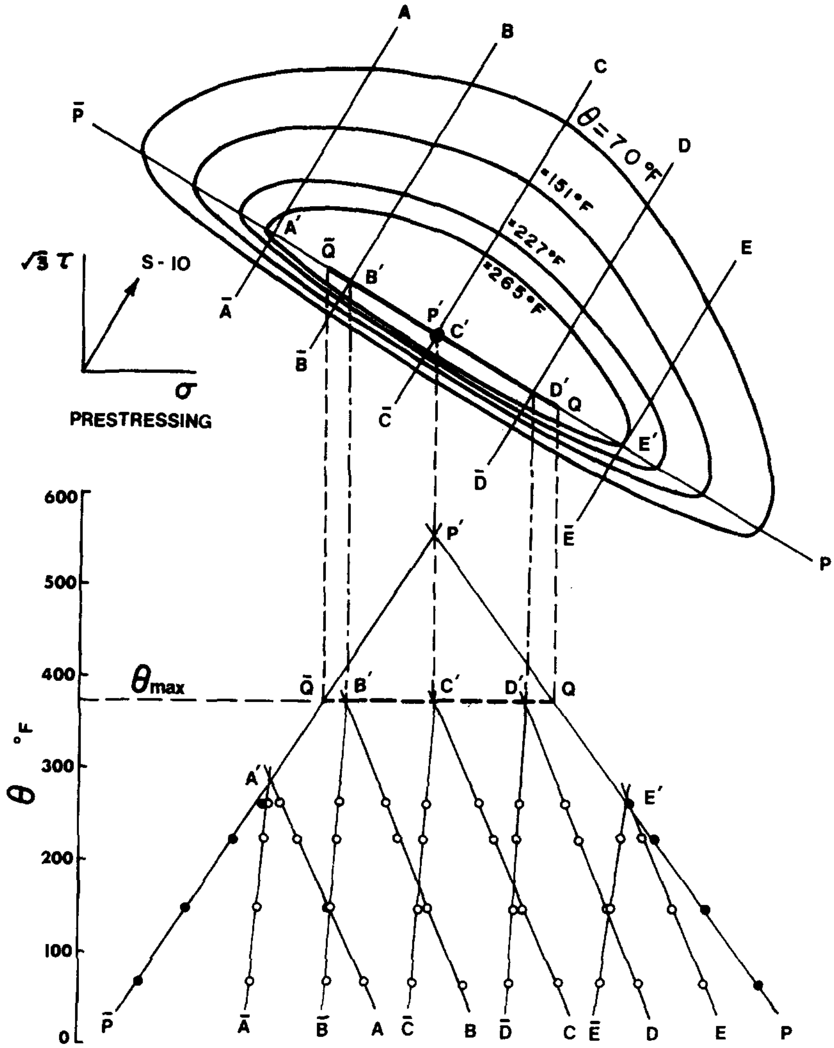


Fig. 8. Experiment *S*-10. First prestressing.

direction. The slope is minimum in the direction of preloading, maximum in the opposite direction. In Fig. 8, lines $\bar{B}B'$, $\bar{C}C'$, and $\bar{D}D'$ have approximately the same slope. The slopes of the lines of the front side of the surface are markedly smaller; $C'C$ has the smallest of the slopes and $B'B$ and $D'D$ have similar slopes. Each of these lines cross the level surfaces at points at which the normals to the preloading direction are to a first approximation parallel. Thus, in accordance with the proposed sandhill model they would be expected to produce the straight lines in the $\theta - \sigma$ plot shown in Fig. 8.

The straight line $\bar{Q}Q$ to which the yield curves degenerate is the familiar ridge line of the sandhill analogy. For the initial yield surface one might expect that the sandhill would be more or less isotropic. Thus, for the Mises surface, plotted in terms of the deviatoric axes in which the Euclidean norm is defined, we would expect that the ridge line would shrink to a point at the apex of a circular cylindrical cone. Our experiments tend to show that for monotonic radial loading the length of the ridge line increases with plastic strain (see Fig. 8).

The next prestressing of the same specimen *S*-10, shown in Fig. 9 is no longer a radial one. This experiment indicates that, with the exception of the ridge line length, the qualitative features observed for monotonic loadings are replicated for more general loadings.

We recapitulate our findings that at each prestressing there exists a straight line perpendicular to the temperature axis which is the limiting yield curve. The limiting yield curve occurs at a limiting temperature. At this stage we shall present our results in the $\bar{\sigma} - \bar{\epsilon}^p$ space introduced in the previous section.

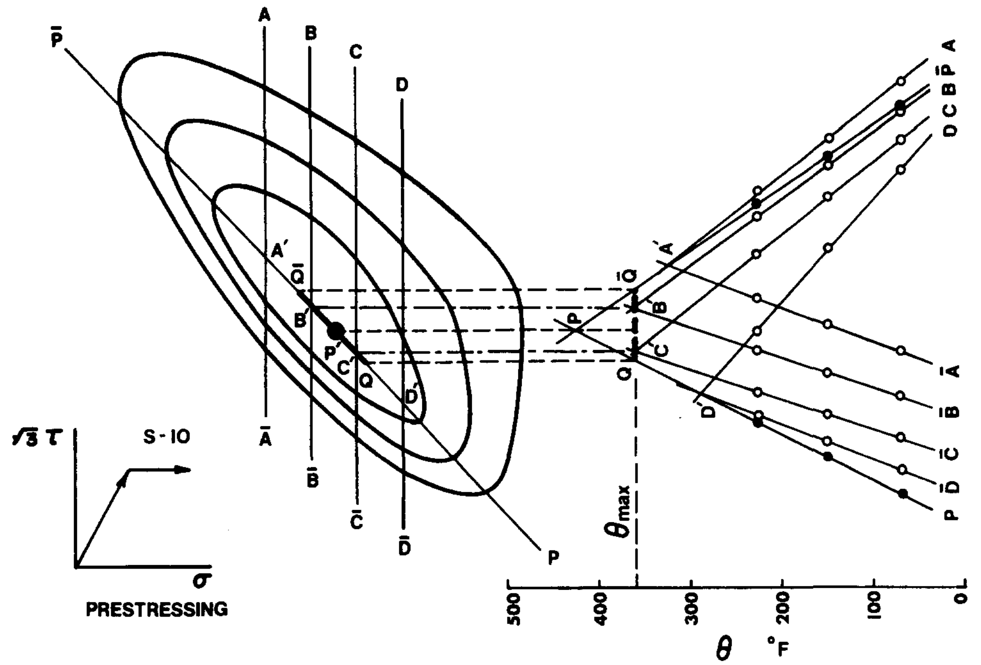


Fig. 9. Experiment $S$-10. Second prestressing.

For the tension–torsion experiments the Euclidean norm defined by eqn (54) leads to

$$\bar{\sigma} = [\sigma^2 + 3\tau^2]^{1/2}$$

$$\bar{\epsilon}^p = \left[\{\epsilon_1{}^p\}^2 + \frac{1}{3}\{\gamma_{12}^p\}^2\right]^{1/2}$$

where

$$\epsilon_2{}^p = \epsilon_3{}^p = -\frac{1}{2}\epsilon_1{}^p.$$

Figure 10 in a double logarithmic scale shows the upper and lower equilibrium stress–strain lines at 70°F. We see that the upper equilibrium stress–strain line for the indicated range of data is a straight line which can be represented by

$$\frac{\bar{\epsilon}^p}{\bar{\epsilon}_y} = 0.004\left[\frac{\bar{\sigma}}{\bar{\sigma}_y}\right]^{7.2}$$

where $\epsilon_y = \sigma_y/E$ is the strain at the room temperature yield stress $\sigma_y$.

The lower equilibrium stress–strain line approaches the upper equilibrium stress–strain line as $\bar{\epsilon}^p$ increases. These two experimentally determined lines should be compared with lines $AC$ and $\overline{AC}$ of Fig. 4. The width of the yield surface in the direction of prestressing decreases as the plastic strain increases. The widths of the yield surfaces in the direction of prestressing as a function of $\bar{\epsilon}^p$ are given in Fig. 10 for each testing temperature. It is seen that the slope of the change in width decreases as the temperature increases with plastic strain remaining constant. Hence, the rate of decrease in width is higher at hign temperatures than at lower temperatures.

The maximum allowable temperature $\theta_{max}$ (°F) starts from approximately 650°F at $\bar{\epsilon}^p = 0$ which is the annealing temperature and decreases according to the formula, for $\bar{\epsilon}^p/\epsilon_y > 1/10$

$$\theta_{max} = 893\left[\frac{\bar{\epsilon}^p}{\epsilon_y}\right]^{-0.038}.$$

In the double logarithmic representation it is a straight line.

The size of the limiting yield surface, that is, the length of the straight line to which the yield surface degenerates at $\theta_{max}$, increases with $\bar{\epsilon}^p$. As $\bar{\epsilon}^p$ approaches zero we would expect the

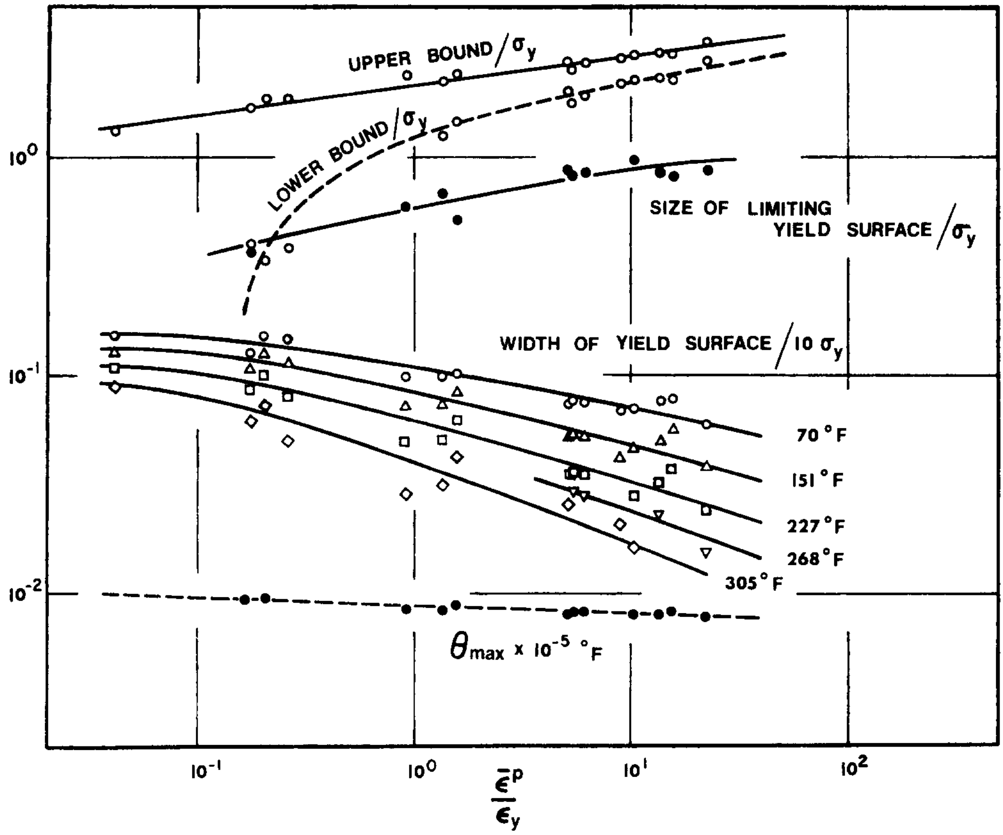


Fig. 10. Experimental results.

ridge line on the high temperature yield surface to degenerate to a point. The positive slope and slight negative curvature of the experimentally determined line shown in Fig. 10 are compatible with this hypothesis.

Figure 11 gives in logarithmic scale the relation between the magnitude of the thermodynamic reference stress $\bar{\sigma}_0$ and $\bar{\epsilon}^p$. We observe that we can express this relation by

$$\frac{\bar{\sigma}}{\sigma_y} = \ln\left(3.763\left[\frac{\bar{\epsilon}^p}{\epsilon_y}\right]^{0.396}\right).$$

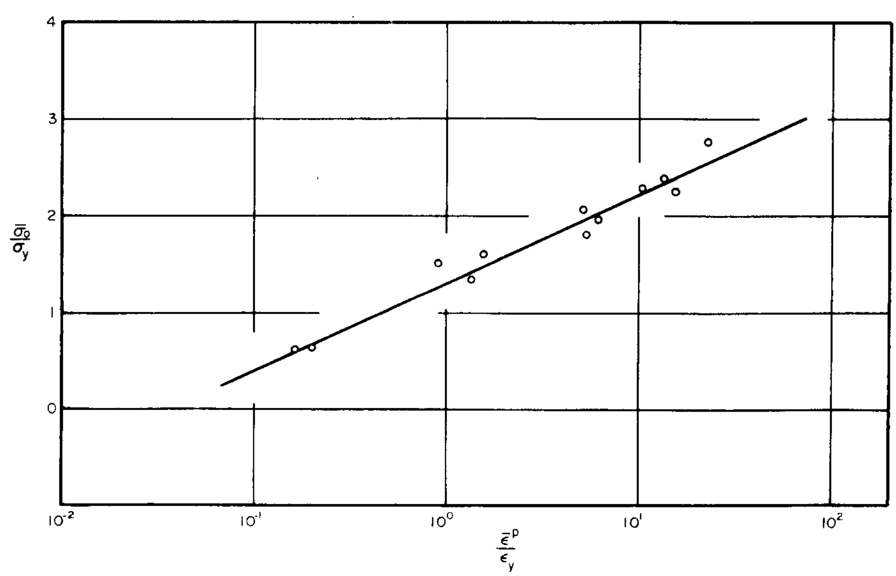


Fig. 11. The relation between $\bar{\sigma}_0$ and $\bar{\epsilon}^p$.

## 5. CLOSURE

The Coleman–Gurtin thermodynamics has been employed to study the properties of a class of thermoplastic materials in which the evolution equation for the internal variables is given by eqn (8) in which thermally activated dislocation motion is neglected. Although eqn (8) is a rate-independent relation between $\dot{k}^{(i)}$ and $\dot{\epsilon}_{ij}''$ the theory is applicable to both rate-dependent and rate-independent materials since no particular law for the growth of $\dot{\epsilon}_{ij}''$ has been assumed.

The most general form (eqn 27) of the Helmholtz potential consistent with the assumption of insensitivity of the elastic relations to inelastic deformation has been derived. From this relation a geometric interpretation of the Clausius–Duhem restriction has been made in stress-temperature configuration space by employing the concept of a thermodynamic reference stress. Under some simplifying assumptions a quantitative experimental evaluation of the relation between $\sigma_{ij}^0$ and inelastic strain has been obtained from the data of Phillips and Tang[3].

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